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Estimation of the volatility parameter in Value at Risk (VaR) model

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Master's thesis

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Abstract. In financial analysis, one of the most commonly used measures for evaluation of market risk is Value at Risk (VaR). Although it is an intuitively simple measure, estimating the underlying volatility can be quite complex. The main objective of the paper is to use Basic, EWMA and GARCH models for volatility parameter estimation in the Value at Risk (VaR) model. Using the real financial data, the methods are compared and determined which one gives the most appropriate estimate of the actual risk. The comparison of models is based on the analysis of the violation process. The results show that there is no single best method - the best model depends on the data to be modelled.

Keywords: Value at Risk (VaR) model, EWMA, GARCH

CERCS: P160 Statistics, operations research, programming, actuarial mathematics.

Volatiilsuse hindamine Value at Risk (VaR) mudelis.

Magistritöö

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Lühikokkuvõte. Finantsanalüüsis on üks kõige sagedamini kasutatavaid vahendeid tururiski hindamiseks nn riski all olev väärtus (VaR - Value at Risk). Ehkki see on intuitiivselt lihtne mõõdik, võib selle aluseks oleva volatiilsuse hindamine olla üsna keerukaks. Töö peamine eesmärk on kasutada volatiilsuse hindamiseks peale tavameetodi ka EWMA ja GARCH mudeleid. Kasutades Riia börsi andmeid, on meetodeid omavahel võrreldud ja välja selgitatud, milline neist annab tegelikule riskile kõige adekvaatsema hinnnangu. Mudelite võrdluse aluseks on rikkumisprotsessi analüüs, kus mudeli abil arvutatud VaR-i võrreldakse tegelike kadudega. Tulemused näitavad, et ühtainsat parimat mudelit ei ole - parim mudel sõltub modelleeritavatest andmetest.

Märksõnad: Value at Risk (VaR) mudel, EWMA, GARCH

CERCS: P160 Statistika, operatsioonianalüüs, programmeerimine, finantsja kindlustusmatemaatika.

Contents

ABBREVIATIONS

V_t - value of asset at time t

r_t - return at time t

α - confidence level

L, l - loss

μ_t - conditional expectation of the return r_t

σ_t - conditional standard deviation of r_t

ϵ_t - shock in financial market or relevant markets

γ_m - m-th lag autocovariance

ρ_m - m-th lag autocorrelation

$MA(q)$ - Moving Average model with q parameters

$AR(p)$ - Autoregressive model with p parameters

$ARMA$ - Autoregressive Moving average model

ACF - autocorrelation function

$PACF$ - partial autocorrelation function

AIC - Akaike Information Criterion

BIC - Bayesian Information Criterion

$EWMA$ - Exponentially - Weighted Moving Average model

λ - smoothing parameter in EWMA model

$GARCH$ - General Autoregressive Conditionally Heteroscedastic model

VR - Violation Ratio

iid - independent identically distributed

1 Introduction

The techniques used to measure risk are of crucial importance for financial institutions - insurers, banks, investment funds and others. An improperly estimated risk may have a negative effect on profitability and financial stability.

In 1996, the Basel Committee on Banking Supervision at the Bank for International Settlements had imposed to use Value at Risk to estimate, control and manage risk. VaR has now become the standard measure used to quantify market risk. [9, p.5]

VaR determines the potential maximal loss over a fixed time period at a allowed probability of occurrence. The key to estimate VaR is to estimate the distribution of losses. Although it seems an easy task, when analysing real financial data, it is a challenging statistical problem. Instead of a loss distribution, the returns distribution is usually modelled.

The main objective of the thesis is to use Basic, EWMA and GARCH models for volatility parameter estimation in the Value at Risk (VaR) model. Using the real financial data, the methods are compared and determined which one gives the most appropriate estimate of the actual risk. Freelance software R $3.6.3$ is used to build and compare models.

The thesis is organized as follows: the first introductory section is followed by a literature review on the topic. First, the definition and properties of risk are recalled, as well as the main definitions related to returns, followed by the definition of VaR, the properties and the basic scheme of the estimation process. Section 2 also includes a definition of the Basic model of returns, from which more complicated models are derived, including models considered in the thesis - EWMA and GARCH models, the explanation of these models follow the description of the Basic model. The section concludes with an outline of methods for comparing different models that measure the accuracy of VaR estimation. Section 3 describes the application of Basic, EWMA and GARCH models in the estimation of VaR for real data sets. The first part of the section contains a description of the data used in the analysis, followed by an outline of the process for estimating VaR using the models. The section is concluded with a summary of obtained results, result analysis and conclusions. The main conclusions are presented in section 4. Resulting graphs of all data sets and R code are included in the appendixes. The thesis consists of 44 pages and contains 5 tables, 6 figures and 2 appendices.

2 Value at risk and its estimation methods

This section summarizes the theoretical basis of Value at Risk and its estimation techniques using various models. Also basic definitions of risk, returns and time series tools used in modelling are included.

2.1 Risk and return

The risk is a key term, a phenomenon that is measured by VaR. Different definitions of risk can be found in various sources, depending on the context of the material. The following definition of risk is proposed in this thesis.

Definition 1. ***(Risk)*** *[9, p.6] The risk is the possibility of losses due to unexpected outcomes caused by financial market movements.*

The risk is classified into five types based on the cause of the potential loss. [9, p.6]

1. Credit risk - potential losses incurred if a counterparty defaults.
2. Operational risk - potential loss due to transaction and payment errors, including fraud and regulatory risks.
3. Liquidity risk - potential loss caused by an unexpected, large, negative cash flow in a short period of time.
4. Market risk - potential loss caused by changing circumstances in the market.
5. Model risk - potential loss caused by a incorrectly specified risk measurement model. [6, p.327]

The measure of interest - the Value at Risk, classically measures market risk, although adaptable to other risks. Only market risk evaluation is considered in the thesis. [9, p.6]

The simplest measure of risk is unconditional variance or unconditional standard deviation. The variance (standard deviation) measures the absolute deviation from the expected mean value, but does not determine whether the change is positive or negative, so it is not possible to distinguish profit and loss, which is crucial aspect in financial data analysis. Unconditional variance does not take into account the time order of observations and previously available information, so the use of time-dependent standard deviation or volatility should be considered. [6, p.331]

The returns of the value of an asset plays a crucial role in risk estimation.

Definition 2. ***(Return)*** *[10, p.255] Return or relative gain of an asset at time t is*

$$r_t = \frac{V_t - V_{t-1}}{V_{t-1}}, \tag{2.1}$$

where V_t is the value of an asset at time t.

It follows from the formula (2.1) that

$$V_t = V_{t-1}(1 + r_t). \tag{2.2}$$

It is now possible to define profit and loss using the value of an asset and returns. If profit is defined as

$$Pr_t = V_{t-1} r_t, \tag{2.3}$$

the loss can be defined as

$$L_t = -Pr_t = -V_{t-1} r_t. \tag{2.4}$$

In practice, values of asset data are usually close to a random walk process and are non-stationary, which makes difficulties to apply some time series models to data modelling. Since returns are commonly consistent with stationarity assumptions, return paths are studied and changes in returns are of interest rather than actual values of assets. [6, p.7]

It follows directly from formula (2.4) that the loss depends only on the returns and with the stationarity effect, it indicates that the study of returns provides a complete overview of the risk.

To evaluate the risk of gaining a "worst-scenario" loss using return data, the aim, is to find a return r^* such that the actual return r_t is most likely t5o be larger than r^*. [10, p.255-256]

2.2 Value at Risk

As mentioned above, one of the simplest measures of risk is volatility, but as volatility in financial markets has increased significantly in recent decades, a more advanced risk measurement tool is needed. Value at Risk (VaR) has become the standard measure for quantifying market risk. [9, p.6]

Definition 3. ***(Value at Risk (VaR))*** *[3] Given the confidence level $\alpha \in (0, 1]$, the Value at Risk (VaR) of the portfolio at the confidence level α is a smallest number l such that the probability that the loss L in specific time horizon exceeds l is not larger than $(1 - \alpha)$. Formally*

$$VaR_\alpha = \inf\{l \in R : P(L > l) \leq 1 - \alpha\} = \inf\{l \in R : F_L(l) \geq \alpha\}, \tag{2.5}$$

where F_L is the distribution function of loss.

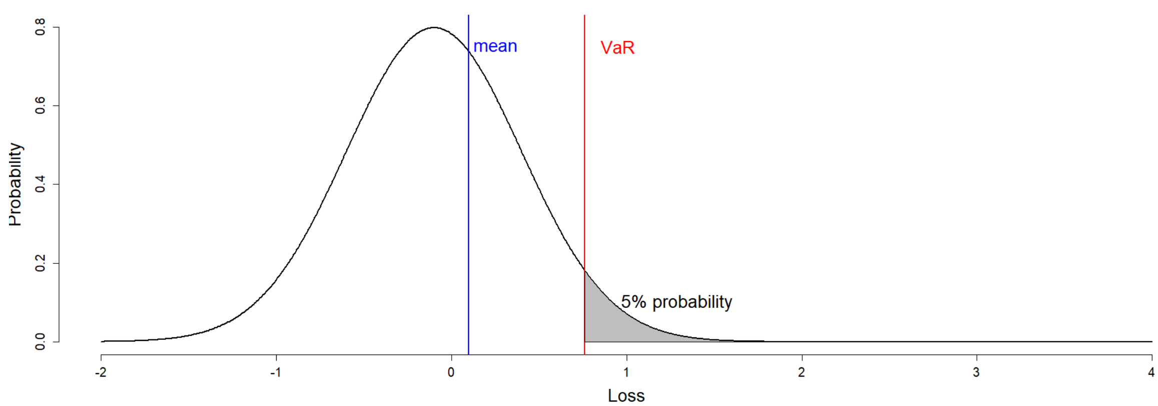

Figure 2.1: *Graphical interpretation of Value at Risk*

Intuitively, this is the maximum potential loss for a given probability in a particular period of time. Statistically, VaR is an estimation of the loss distribution quantiles [9, p.6-7], it is also shown graphically in the figure 2.1.

Although the concept of VaR intuitively is a very simple measure, to evaluate VaR can be mathematically challenging task. Various sources suggest different methods for estimating VaR, but all methods have a common general structure: [9, p.8]

1. Portfolio risk mapping
2. Estimation the loss distribution of the portfolio
3. Computation of the VaR

The main differences between the distinct methods are related to the estimation of loss distribution. Results using different methods can vary more than 14 times, so before choosing the method, should take into account the type and characteristics of the portfolio, as well as the underlying assumptions of the model. [9, p.8]

In practical application, the estimation of VaR involves making decisions on following. [11, p.258]

1. α value selection. The most commonly used α values are 90%, 95%, 99% and 99.9%. A value of $1-\alpha$ indicates how many observations out of 100 are expected to exceed the VaR.
2. Time horizon selection. The choice of period depends on how long a forecast is needed. VaR generally works better for short-term forecasts, the longer-term forecasts need to be made, the longer the time period to be included.
3. Frequency of observations. Daily data are usually used in the analysis of price data, mainly because hourly data are not stored or are more expensive.
4. Selection of the cumulative loss distribution function. Depending on the method, different distributions can be used for estimation. In practice, distribution of returns is commonly used instead of loss distribution. This approach is also used further in the thesis.

5. Market-to-market value of the portfolio, in other words, the amount of the financial position. If several assets are included in a portfolio, it is an essential part of the process.

Despite easy intuitive interpretation, a wide range of estimation methods, and widespread use in practice, VaR is still sometimes criticized and some alternatives are sought because of limitations and shortcomings of it. As mentioned above, VaR models have a short-term focus, in other words, they provide imprecise estimates for long-term forecasts. VaR only estimates losses related to the effects of market risk (although there are methods to adjust VaR to other risks, it does not combine all types of risk together). Another disadvantage, unlike alternative measures (such as the Expected Shortfall), the VaR does not provide any information on the volume of losses above the calculated VaR. It is important to note that all VaR methodologies use past observations in some degree. If there is only data for a stable period in the selected time horizon, the VaR may be underestimated, if it contains data for a fairly volatile period, the VaR may be overestimated. Moreover, all methodologies make some assumptions about the distribution of losses, which, if not appropriate, lead to incorrect VaR calculations. This leads to the previously stated conclusion that the choice of distribution (model) is of great importance in the VaR assessment process. [2, p. 408]

2.3 Basic model of returns

The Basic model of returns describes how returns can be expressed using returns distribution characteristics.

Following [4, p.7], to model the returns $r_1, r_2, ...$ of an underlying asset, it is known that the model can be created using the formula

$$r_t = \mu_{t-1} + \sigma_{t-1}\epsilon_t, \tag{2.6}$$

where

$\mu_t = E(r_t|F_{t-1})$ - conditional expectation of the return r_t, on the information available at time $t-1$,

$\sigma_t = Var(r_t|F_{t-1})$ - conditional standard deviation of the r_t, on the information available at time $t-1$,

ϵ_t - shock with a conditional mean equal to 0 and a conditional deviation equal to 1 (usually assumed to be iid).

The values μ and σ are not observable and must be estimated. The simplest approach to estimating μ and σ is to use the classical unconditional mean and variance estimates $\hat{\mu}$ and $\hat{\sigma}^2$ according to formulas (2.7) and (2.8), respectively.

$$\hat{\mu} = \frac{1}{T-1} \sum_{j=1}^{T-1} r_j, \tag{2.7}$$

$$\hat{\sigma^2} = \frac{1}{T-1} \sum_{j=1}^{T-1} (r_j - \hat{\mu})^2. \tag{2.8}$$

The disadvantage of this approach is that this estimation is unconditional - it does not take into account the time order of observations and assigns equal weight to all observations, although intuitively more recent information should be more relevant. This means that a simple estimate can only be used as a threshold or a benchmark, but better estimation methods should be considered. [1, p.28]

When using the time series approach in returns modelling, the most important parameter to be estimated is volatility (the mean of returns is relatively small, approximately 0). The main difficulty is that volatility is not directly observable from the data provided. For example, considering the daily data of a stock, only one observation is made per day, so it is not possible to observe daily volatility. [11, p.80] Although not directly observable, there are some features of the volatility that can be observed by analysing the data. [11, p.80]

1. Volatility clustering. Periods of low volatility are followed by periods of high volatility and vice versa.

2. The evolution of volatility has a continuous manner over time, in other words, outliers are relatively rare.

3. Volatility is finite and varies over some fixed time interval. This property indicates that a simple variance estimate (2.8) should be replaced by a more sophisticated one.

4. Leverage effect - stock prices react more sharply to negative market shocks (for example, declining stocks of affiliated companies, negative changes in gold prices, international developments, etc.) - changes are faster and usually larger (volatility is higher) than in positive shocks.

2.4 Basic terms of time series analysis

When treating returns as random variables over time, time series models can be used to estimate the model. This subsection summarizes some of the basic tools of time series theory from [11, p.23-37]

The most important characteristics of time series that must be met in order to use a wide class of models are stationarity conditions.

Definition 4. ***(Strict stationarity)*** *Time series $\{r_t\}$ is said to be strictly stationary if for all arbitrary positive integers k and for all time moments t joint distribution of $(r_{t_1}, ..., r_{t_k})$ and $(r_{t_1+t}, ..., r_{t_k+t})$ are identical.*

In practice, strict stationarity conditions can rarely be met, a weaker version of stationarity is usually assumed.

Definition 5. ***(Weakly stationarity)*** *Time series $\{r_t\}$ is said to be weakly stationary if*

$$E(r_t) = \mu \tag{2.9}$$

for all time moments t and for all arbitrary positive integers m

$$Cov(r_t, r_{t-m}) = \gamma_m, \tag{2.10}$$

which means that covariance depends only on m, but not on time t.

For real data in practice, weak stationarity means that the data plot shows constant variation over time around a constant mean value.

The function (2.10) depends only on the lag m, it is called the lag-m autocovariance of r_t. The autocovariance function has two important properties:

1. $\gamma_0 = Var(r_t)$
2. $\gamma_{-m} = \gamma_m$

The autocovariance divided by variance forms a function

$$\rho_m = \frac{\gamma_m}{\gamma_0} \tag{2.11}$$

and is called lag-m autocorrelation of r_t and is correlation between r_t and r_{t-m}. Under weakly stationarity conditions autocorrelation depends only on the lag m.

Definition 6. ***(Autocorrelation function (ACF))*** *The sequence $\rho_0, \rho_1, \rho_2, ...$ is called the autocorrelation function of r_t.*

ACF is one of the most significant tools in time series analysis. ACF describes the linear dynamics of time series.

The most important property of ACF is that in the $MA(q)$ model, the ACF of lag q ACF is not 0, but for all ρ_i, $i > q$ ACF is zero. This property allows to specify the number of parameters to be included in the MA model. Another useful property is that the ACF for all lags is an independent random variables with mean 0 and constant variance, that is, white noise.

Definition 7. ***(Partial autocorrelation function (PACF))*** *The sequence $\phi_{11}, \phi_{22}, ...$ is called partial autocorrelation function of r_t, where $\phi_{11}, \phi_{22}, ...$ satisfies the system of equations*

$$\begin{aligned}
r_t &= \phi_{01} + \phi_{11} r_{t-1} + e_{1t} \\
r_t &= \phi_{02} + \phi_{12} r_{t-1} + \phi_{22} r_{t-2} + e_{2t} \\
r_t &= \phi_{03} + \phi_{13} r_{t-1} + \phi_{23} r_{t-2} + \phi_{33} r_{t-3} + e_{3t} \\
&\dots
\end{aligned} \tag{2.12}$$

Just as ACF can be used to determine the number of possible parameters in an MA model, PACF can be used to determine the number of parameters in an AR model. For the $AR(p)$ model, the PACF of lag p is not 0, but for all $\phi_{jj}, j > p$ PACF is zero. This property allows to specify the number of parameters to be included in the AR model.

In addition to the ACF and PACF functions, various information criteria can help to decide the number of parameters included in the model and the type of model. The most commonly used are the following.

Definition 8. ***(Akaike Information Criterion (AIC))*** *Akaike Information Criterion (AIC) defined by formula*

$$AIC = \log\left(\frac{SSE}{n}\right) + \frac{n+2k}{n} \tag{2.13}$$

where SSE - residual sum of squares under the model with k coefficients and n - number of observations.

Definition 9. ***Bayesian Information Criterion (BIC)*** *Bayesian Information Criterion (BIC) defined by formula*

$$BIC = log\left(\frac{SSE}{n}\right) + \frac{k \log n}{n}. \tag{2.14}$$

2.5 Exponentially - Weighted Moving Average

The outline of this section is based on [1, p.27-35], unless stated otherwise.

In analysing financial data to predict future value based on known historical observations, it seems intuitive that more recent observations provide more relevant information about fluctuations than older ones. It is a reasonable idea to apply weights to observations - the older the observation, the lower the applied weights. This is the main idea of the Exponentially - Weighted Moving Average model.

Definition 10. ***(Exponentially - Weighted Moving Average (EWMA))*** *Given the smoothing parameter $0 < \lambda < 1$, the Exponentially - Weighted Moving Average (EWMA) model assumes that the volatility forecast for the next period in time-varying according to the rule*

$$\sigma^2_{EWMA} = (1-\lambda)\sum_{j=0}^{t} \lambda^j r^2_{t-1-j}. \tag{2.15}$$

Some properties of the method follow from the formula (2.15):

1. The significance of the observation decreases over time at the rate of $(1-\lambda)\lambda^j$.
2. The lower the value of λ, the more recent observations influence the estimation.
3. If $\lambda = 1$, then the formula simplifies to classical variance estimation (2.8).

The maximum likelihood method is usually used to estimate the smoothing parameter λ. Since the model has only one parameter λ to evaluate, it is quite robust in terms of estimation error, compared to other models [8, p.9]. Despite the use of the maximum likelihood method to estimate λ, assumptions about a suitable distribution are needed, which complicates the problem. Studies show that it is usually sufficient to use $\lambda = 0.94$ for daily financial data to avoid estimation of the smoothing parameter issue.

The advantage of the EWMA model in volatility modelling is that it is easy to use, it has only one parameter . The EWMA model is also able to capture non-linear effects, such as clusters - one of the most common characteristics of financial data volatility. [14, p.32]

The disadvantage of the model related to a time horizon longer than 1 is that, since the distribution of the cumulative returns estimations obtained using σ^2_{EWMA} is not known, the VaR estimate can only be obtained using Monte Carlo method, in other words, using simulations. In addition since all historical values are used in the model, extreme values (outliers) have a significant effect on the estimation of volatility over a long period of time.

The EWMA method also does not allow to generate skewed distributions. To improve the EWMA model, GARCH models can be used. Unlike the EWMA model, the GARCH models enable non-Gaussian set-up and skewed distributions.

2.6 GARCH model

Recent studies show that heteroscedasticity (the variance of the forecast error depends on the magnitude of the previous disturbances) is a fairly common phenomenon in financial data. Since conventional ARIMA class models assume homoscedasticity, a wider class of models should be considered, introducing an autoregressive conditionally heteroscedastic (ARCH) model class. [7, p.658]

Recalling the basic model of returns defined by the formula (2.6), it can be assumed that return consists of two parts - the conditional mean and the multiplication of shock value. This multiplication is called the innovation process and can be expressed as the GARCH process.

Definition 11. ***(GARCH model)*** *[6, p.19] The process $\{a_t\}$, where $a_t = \sigma_t \epsilon_t$ is called the General Autoregressive Conditionally Heteroscedastic GARCH (p,q) process if its two first moments exist and $E(a_t|F_{t-1}) = 0$ and exist constants w, β_i, $i = 1, 2, ..., q$ and α_j, $j = 1, 2, ..., p$ such that*

$$\sigma^2_{GARCH} = w + \sum_{i=1}^{q} \beta_i a^2_{t-i} + \sum_{j=1}^{p} \alpha_j \sigma^2_{t-j}. \tag{2.16}$$

The parameters w, β_i, $i = 1, 2, ..., q$ and α_j, $j = 1, 2, ..., p$ can be estimated from the returns data using least squares or maximum likelihood methods. The main assumption of the model is that the standardized residuals are iid random variables. [9, p.8-9]

The choice of the values of the parameters p and q is not very trivial. Values can be achieved using various tools: [6, p.108]

1. autocorrelation and partial autocorrelation functions for the residuals of $ARMA(p, q)$ model $a_1^2, a_2^2, ..., a_n^2$;
2. information criteria;
3. testing the significance of coefficients;
4. residuals analysis.

Tools 2. - 4. are mostly used to validate models and compare different models with different parameter values, while ACF and PACF are used directly to select the values p and q. The logic for choosing p and q is the same as for ARMA model class, based on the ACF and PACF values of the squared residuals of the ARMA type model. [6, p.108-109]

One of the most commonly used models for financial data is the $GARCH(1, 1)$ model (it is even shown that in practice cases where the model with other values of parameter p and q values outperform the $GARCH(1, 1)$ model are rare). According to the general case (2.16), the formula for $GARCH(1, 1)$ is [11, p.94]

$$\sigma_t^2 = w + \alpha a_{t-1}^2 + \beta \sigma_{t-1}^2 \tag{2.17}$$

Using the formula (2.17), some properties of the method can be deducted: [11, p.94-95]

1. Large values of β imply a significant dependency on past volatility. [4, p.21]

2. High shock values a_{t-1} leads to a rise in volatility estimates.

3. The GARCH (1,1) process tails are heavier than the Gaussian distribution tails.

Although the model has some useful properties, such as simple interpretation of parameters, this method also has some drawbacks. Unlike the EWMA model, the GARCH model does not capture clustering and responds equally to positive and negative shocks. The GARCH model tends to over-predict volatility, which can lead to underestimation of VaR. Another important aspect is that since all elements enter quadratic in the GARCH formula, the extreme values of observations can evoke instability in the parameter estimates. [9, p.8-9]

2.7 Comparison of models

As mentioned previously, there is a broad class of models that can be used to estimate VaR. Some of them are described in more detail in the previous subsections. All the models considered have advantages and disadvantages. To choose which model gives a better VaR estimation, some measures need to be introduced to compare the models. As the volatility of daily financial data is not directly observable, it is also challenging to compare the performance of different volatility estimation models.

The disadvantages of comparing models are that there is no general type of comparison, almost all studies use different methods. Typically, several measures are used to decide on the performance of a model.

Some sources suggest comparing the estimated (forecasted) volatility $\hat{\sigma}_{t+h}$ at time $t + h$ with the shock value a^2_{t+h}. However, from a statistical point of view, considering a 1-step ahead forecast, as $E(a^2_{t+1}|F_t) = \sigma^2_{t+1}$, then a^2_{t+1} is a consistent estimate of volatility, but since a random variable is observed per single time unit, it cannot provide an accurate estimate of variance. [11, p.100]

The most common method for backtesting of VaR is the violation process.

Definition 12. ***(Violation process)*** *[5, p.50] From VaR definition (2.5) it follows that the violation process is defined as*

$$I_t(\alpha) = 1_{\{L(t) > VaR_\alpha(L(t))\}}. \tag{2.18}$$

VaR forecasts are valid if and only if the violation process (2.18) satisfies two conditions:

- $E(I_t(\alpha)) = 1 - \alpha$;
- $I_t(\alpha)$ and $I_s(\alpha)$ are independent for all $s \neq t$.

If the conditions are met, each violation is Bernoulli distributed, but the number of violations - binomial distributed. In practice, this means that in the violation process, the actual values of VaR are replaced by estimates and it is checked whether the process behaves like independent identically Bernoulli distributed random variables with a violation probability close to $1 - \alpha$. If the proportion of VaR violations does not differ significantly from $1 - \alpha$, then it can be concluded that the VaR estimate is reasonable. [5, p.51]

Definition 13. ***(Violation ratio)*** *[2, p.408] The violation ratio is defined by formula*

$$VR = \frac{\textit{observed number of violations}}{\textit{expected number of violations}} = \frac{\sum_t^T I_t(\alpha)}{(1-\alpha)T}, \tag{2.19}$$

where T - number of estimates.

It can be said that, if $VR > 1$ then the VaR model under-forecasts risk, but if $VR < 1$ then - over-forecasts. In practice, such strict rules are not used, for practical tasks, the predictions for which $VR \in [0.8; 1.2]$ are considered to be adequate. [2, p.408] Although, if the expected number of violations is small, then even one violation difference between observed and expected violations can lead to too low/high VR value. For example, having $T = 200$ forecasts at a 99% significance level, the expected number of violations is 2. If the observed number of violations is 1, in absolute terms, the estimate could be considered as good, but violation ratio value is low ($VR = 0.5$). To improve this shortcoming, some tests are used to check whether the observed number of violations is statistically equal to the expected number of violations. Most commonly used tests are unconditional coverage test, conditional coverage test and Dynamic quantile test. [14, p.40]

The violation process allows the comparison of models from different classes and with different estimation approaches, it does not require complicated conditions or similar origin of estimations.

3 Application of EWMA and GARCH models in VaR estimation

This section focuses on the practical application of the EWMA and GARCH models in estimation of VaR for a set of stock prices. Achieved estimates are compared using a violation process and unconditional coverage test and it is discussed which model gives a better estimation of VaR.

3.1 Data description

Trading data sets are used for modelling. Data sets contain daily adjusted closing prices of stocks of various companies. A 5-year period is selected (from 01.03.2015 to 29.02.2020). The data source is $NASDAQ\ Baltic$ stock market database, chosen companies that are registered in Latvia. List of all companies whose data are analysed are summarized in the table 3.1.

Observations are available for all data sets for all time moments in the period, without missing values. In the data set $data10$ prices are available only starting from 12.07.2016, therefore a shorter time period is used in the analysis.

Further detailed process of model building and analysis will be described for only one data set - the stock prices of *AS Grindeks* ($data1$). For other data sets, the analysis process is not reported in details, but the final VaR estimates are shown graphically in Appendix 1, the main results of the violation process are included in the table 3.4 and also the general findings and conclusions are stated.

For model building and analysis process freelance software $R\,3.6.3$ is used. The main part of the code added in Appendix 2.

Since the data set contains adjusted closing prices, to start the data analysis, the returns must first be calculated using the formula (2.1).

Table 3.1: *List of companies*

Company	Industry	Abbreviation
AS Grindeks	Manufacture of pharmaceutical products	$data1$
AS Olainfarm	Manufacture of pharmaceutical products	$data2$
AS Latvijas balzāms	Manufacture of beverages	$data3$
AS Latvijas Gāze	Trade of gas	$data4$
AS PATA Saldus	Logging	$data5$
AS Valmieras Stikla šķiedra	Manufacture of glass fibers	$data6$
AS Rīgas kuģu būvētava	Building of ships	$data7$
AS SAF Tehnika	Manufacture of communication equipment	$data8$
AS Rīgas juviliertizstrādājumu rūpnīca	Manufacture and trade of jewellery	$data9$
AS HansaMatrix	Manufacture of loaded electronic boards	$data10$

Both data sets - price and returns - are shown in figure 3.1. The plot on the left visualizes the adjusted closing prices and the plot on the right plots the return data. Inspecting the price plot, as the mean of the prices is not constant, it can be concluded that the price data series cannot be stationary. The mean value of returns, on the other hand, look fairly constant and small - around 0. The variance of returns has a constant manner, with some local peaks occurring. These shocks cause short-term high fluctuations of returns, followed by rather smooth, small fluctuation period - this is called clustering and it is one of the characteristics of volatility.

VaR estimates at each time point t are calculated based on data of the previous 250 time moments $t-250, t-249, ..., t-1$ (approximately 1-year data in trading days). In figure 3.1, the data set used to estimate the first VaR value (251st time moment) is separated by a red line. Moving on, the window used in the modelling is moved one step forward at a time.

The next step is to start building models. The returns of adjusted closing prices are used

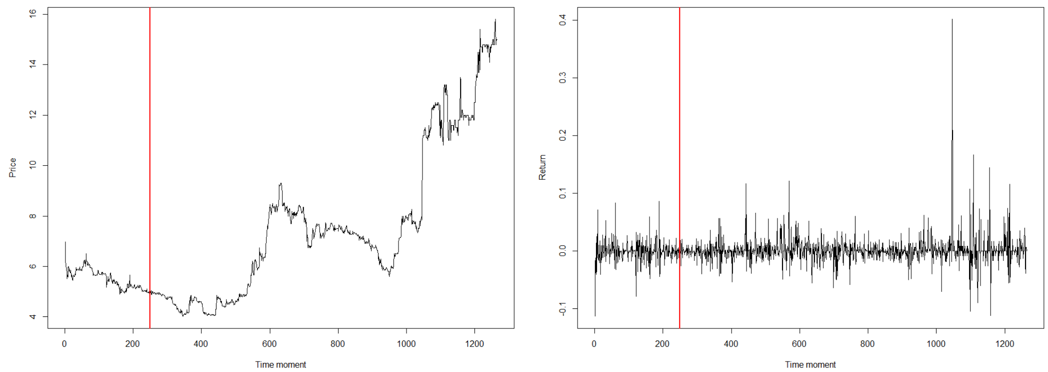

Figure 3.1: *Data set* $data1$ *adjusted closing prices and returns. On the left - adjusted closing price plot, on the right - adjusted closing price returns.*

in the modelling process.

3.2 Volatility estimation using basic model

This estimation is based on a simple returns model (2.6), assuming that the shocks have a standard normal distribution and are iid and the mean and variance estimates are calculated using classical formulas - (2.7) and (2.8).

This estimate is used as a benchmark only. It is known that the modelling process can be costly, this estimate helps to distinguish whether the use of a more complex model provides a significant improvement in VaR estimation.

The procedure is as follows, at each time moment $k = 251, 250, ..., 1264$:

1. Fix the data set window that will be used in the k-th step: $r_{k-250}, r_{k-249}, ..., r_{k-1}$.
2. Using the formula (3.1), calculate VaR estimate at each time period

$$VaR_{\alpha,k} = \mu_k - \sigma_k \epsilon_{k,\alpha}, \tag{3.1}$$

where

(a) σ_k - volatility calculated using the classical variance formula (2.8)

(b) μ_k - the unconditional mean value is used, calculated by formula (2.7)

(c) shock ϵ_k is assumed to be α-quantile of the standard normal distribution.

To compare various significance levels, VaR is calculated at two significance levels - 95% and 99%.

The Basic model does not take into account the time order of observations in the estimation window, although intuitively more recent observations may provide more valuable information than the older ones. The EWMA model is used for this approach.

3.3 Volatility estimation using EWMA model

The process of creating an EWMA model is simple - there is no need to choose the number of parameters and even estimate them. As stated before, the value of the smoothing factor is usually used as $\lambda = 0.94$.

For ease of use in the further VaR estimation process, the classical EWMA formula (2.15) can be reduced to

$$\sigma^2_{EWMA} = \lambda\sigma^2_t + (1-\lambda)r^2_{t-1}. \tag{3.2}$$

To use the EWMA model in VaR analysis, at each time moment $k = 251, 252, ..., 1264$: [6, p.335]

1. Fix the data set window that will be used in the k-th step: $r_{k-250}, r_{k-249}, ..., r_{k-1}$.
2. Assuming that σ^2_k follows the EWMA model formulated by (3.2), estimate σ_k - the volatility at time moment k .
3. Using the formula (3.1), calculate VaR estimate at each time period, where

   (a) σ_k - the volatility estimated in the second step

(b) μ_k - the unconditional mean value is used, calculated by formula (2.7)

(c) For the EWMA model according to the model specification, it is assumed that shock ϵ_k is the α-quantile of the standard normal distribution.

To compare various significance levels, VaR is calculated at two significance levels - 95% and 99%.

So far, estimates have been achieved using Basic and EWMA models. Both models assume that shock values are normally distributed, although normality in real data sets are rather rare, so a more advanced approach is used in the GARCH model.

3.4 Volatility estimation using GARCH model

To begin building the GARCH model, it is necessary to evaluate the autoregressive or linear effects in the time series of returns to model the mean with the purpose to obtain residuals of zero mean (in other words, to separate innovation process). If not, the mean value dynamics will affect the variance estimation and will not be distinguished. The properties of ACF and PACF described in subsection 2.4., show how to determine possible values of the parameters p and q of $ARMA(p,q)$ model.

The ACF and PACF diagrams for the data set $data1$ are shown in the figure 3.2. As both graphs show a slow decay in ACF and PACF, and the first lag in ACF and PACF is significant, the possible models are $AR(1)$ and $MA(1)$. As several sources recommend to use $GARCH(1,1)$ in volatility modelling, also $ARIMA(1,1)$ is checked.

If there are several possible models, the one with the lowest AIC and / or BIC values is preferred. In ambiguous cases, the model with the smallest number of parameters is chosen. The criteria values of candidate models are gathered in the table 3.2, the lowest values of both criteria are for the model $MA(1)$, it can be considered as the best model.

The same p and q values are used for to build the GARCH model, as input data set use the squared residuals of $ARMA(p,q)$ model. The $GARCH(0,1)$ model is used to model the volatility of the data set $data1$.

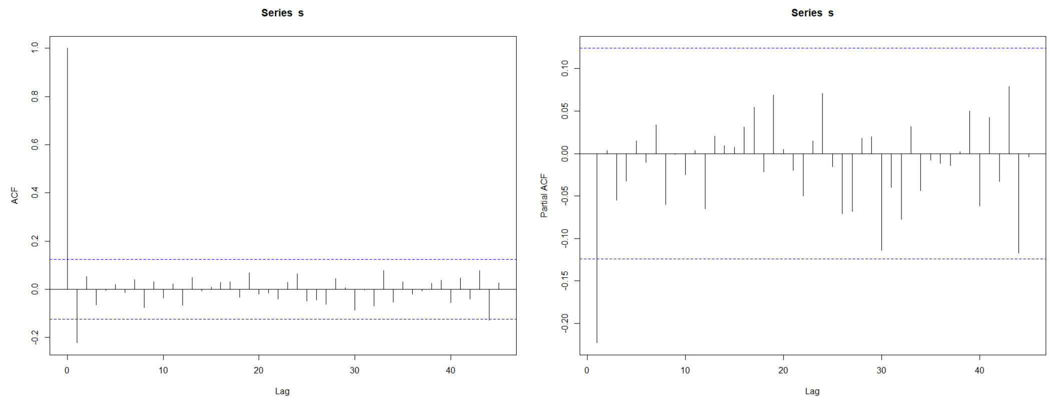

Figure 3.2: *ACF (on the left) and PACF(on the right) plots of returns in data set* $data1$

Table 3.2: *Comparison of ARMA models considered to evaluate linear effects in data set* $data1$

Model	AIC	BIC
$AR(1)$	-1 263.258	-1 252.964
$MA(1)$	-1 263.544	-1 252.979
$ARMA(1,1)$	-1 261.606	-1 247.520

Before getting to the actual estimation process, the model assumptions should be tested. The key assumption of the GARCH model is that the standardized residuals are iid random variables, so residuals need to be tested to use this model in the further estimation process. To perform residuals diagnostics, the diagnostics plots are examined - the residuals of GARCH model plot, the residual ACF plot and the Box-Ljung test values. ACF of residuals is the most important graph, there should be no effect left in the residuals. The normality of residuals is not mandatory. The Box-Ljung test determines whether the residuals are correlated.

The diagnostics graphs of the sample data set $data1$ are plotted in figure 3.3. As can be seen, there are no significant relations in residuals. As the assumption of the GARCH model is fulfilled, the estimation will be performed using the $GARCH(0,1)$ model.

To use GARCH model in the VaR analysis at each time moment $k = 251, 252, ..., 1264$, given that $(a_1^{(250)})^2$, $(a_2^{(250)})^2$, ..., $(a_{250}^{(250)})^2$ is the data set used in the $GARCH(0,1)$ model,

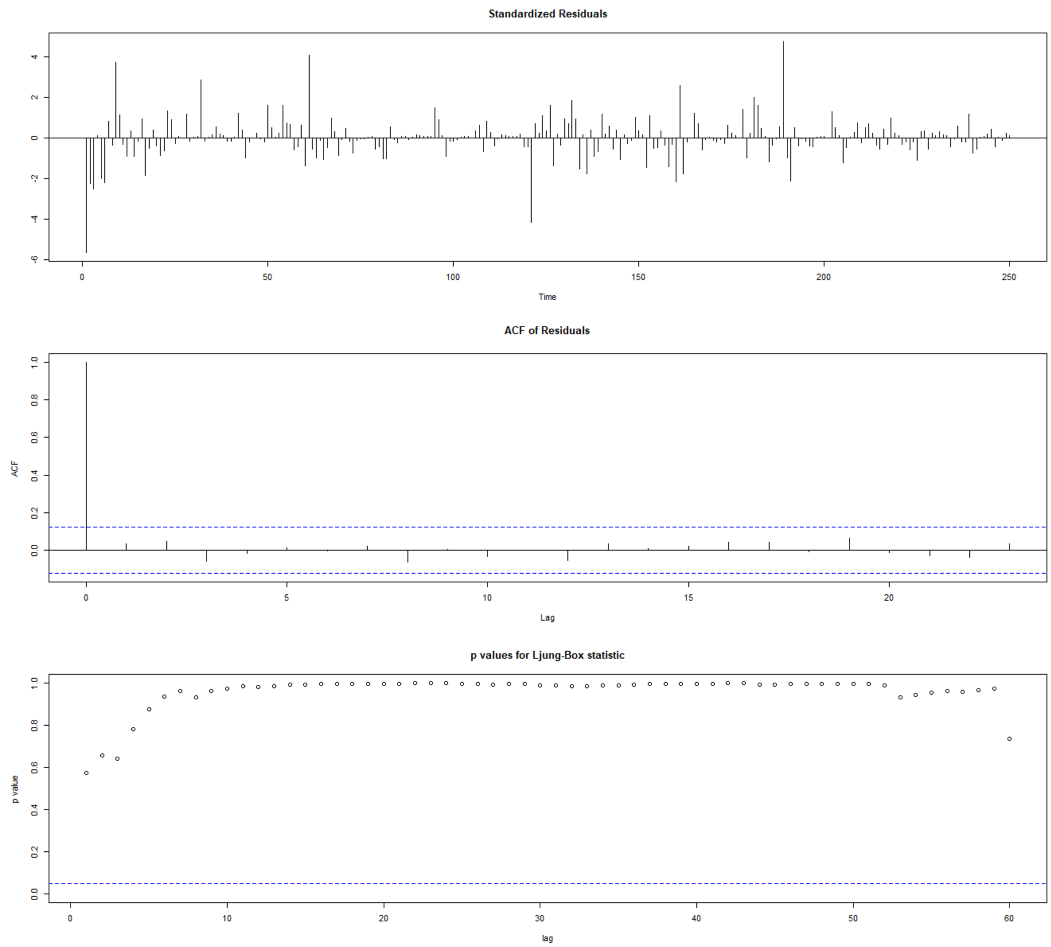

Figure 3.3: $GARCH(0, 1)$ *model residuals plot (upper), ACF plot of residuals (middle) and p-values of Box-Ljung statistics plot (lower)*

proceeding as follows: [6, p.335]

1. Fix the data set window that will be used in the$k$-th step $r_{k-250}, r_{k-249}, ..., r_{k-1}$. This data set is used to generate $ARMA(0, 1)$ model.

2. Find the residuals of the $ARMA(0, 1)$ model $a_1^{(k)}, a_2^{(k)}, ..., a_{250}^{(k)}$.

3. Assume that the σ_k^2, follows the $GARCH(0, 1)$ model formulated by (2.16). The squared residuals $(a_1^{(k)})^2, (a_2^{(k)})^2, ..., (a_{250}^{(k)})^2$ are used in the $GARCH(0, 1)$ model to estimate σ_k - the volatility at time moment k.

Note that the type of GARCH (parameter values of p and q) remains the same as previously modelled, but the coefficients are re-estimated in each step k.

4. Using the formula (3.1), calculate VaR estimate at each time period, where

   (a) σ_k - the volatility estimated in step 3

   (b) μ_k - the unconditional mean value is used, calculated by formula (2.7)

   (c) shock ϵ_k is an α-quantile of the distribution F.

   The distribution F in the practical tasks is unknown and needs to be estimated. A simple non-parametric estimate suggests assuming F to be the empirical distribution of the standardized residuals achieved in the chosen model.

To compare various significance levels, VaR is calculated for two significance levels - 95% and 99%.

The evaluation of the VaR using GARCH, EWMA and Basic models has been obtained, now continue with model comparison.

3.5 Comparison of the results

For a more detailed look on estimation process at one time moment, the numerical results achieved in step $k = 1$ are presented in the table 3.3. In terms of volatility estimation, EWMA and GARCH models give two times smaller volatility estimate than the classical unconditional standard deviation formula used in the Basic model. Empirical shock values estimation gives significantly higher shock values than those achieved by assuming normally distributed shocks. By plotting the histogram of standardized residuals of ARMA model in figure 3.4, it can be seen that the empirical distribution, that is used to estimate shock in GARCH model, has higher peak than the normal distribution, the mean of it is 0.087, but conditional standard deviation is 1.168.

Comparing the VaR estimates, there are no violations in a particular time moment. The basic model estimates the largest possible loss, while the EWMA model - the smallest.

Table 3.3: *Numerical values of parameters achieved at first estimation step* $k = 1$

	Basic model	$EMWA_{\lambda=0.94}$	$GARCH(0,1)$
μ_{251}	-0.001	-0.001	-0.001
σ_{251}	0.020	0.010	0.010
$\epsilon_{251,95\%}$	1.645	1.645	2.878
$\epsilon_{251,99\%}$	2.326	2.326	4.457
$VaR_{251,95\%}$	-0.034	-0.017	-0.028
$VaR_{251,99\%}$	-0.047	-0.024	-0.043
r_{251}	0.018	0.018	0.018

However, the results of just one step do not provide an overall insight into the performance of the model, only comparing the results achieved for the whole data set a reliable decisions can be made. For visual comparison, it is useful to plot the observed values and the estimated VaR values in one graph. Examples of diagrams for the data set $data1$ are included in the figure 3.5. Plots of other data sets are gathered in Appendix 1.

The violation process is used to compare the estimates with the actual observations to obtain measurable results. To calculate the violation ratio using the formula (2.19), the

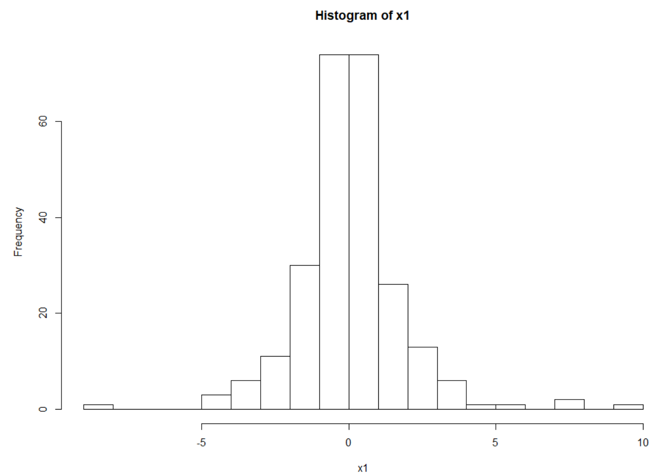

Figure 3.4: *Histogram of standardized residuals of $ARMA(0,1)$ model in data set $data1$ at step $k = 1$. Mean is 0.087 and standard deviation is 1.168*

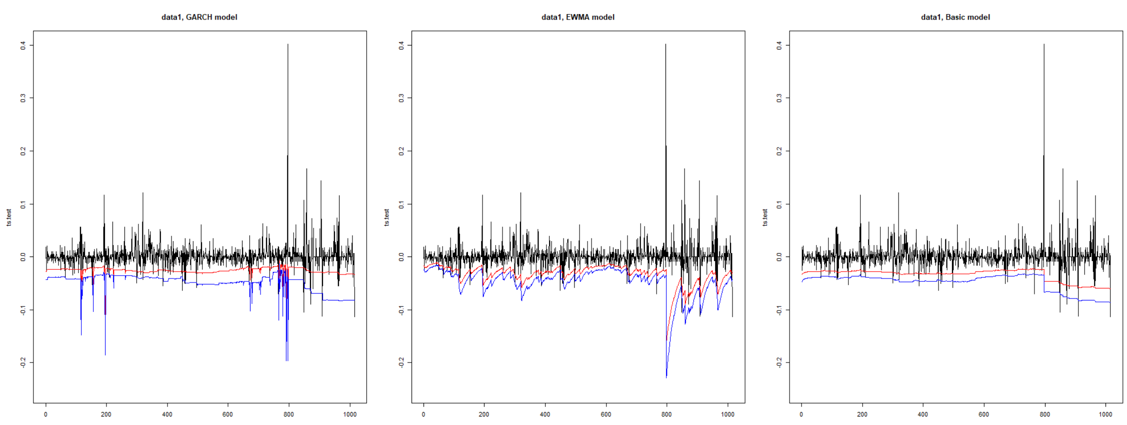

Figure 3.5: *$data1$ observed values of returns vs. estimated VaR values. On the left - GARCH model, in the middle - EWMA model and on the right Basic model. Black line - observed values, red line - estimated values at a 95% significance level and blue - at 99% significance level*

number of observations that are smaller than estimated VaR values are counted and compared with the expected number of observations exceeding the VaR. Returning to VaR definition in section 2.2, the expected number of observations exceeding VaR estimate is $(1-\alpha)$% of number of observations. In the analysed data sets, the number of observations is 1015, so the expected number of violations at 95% significance level is 50.75 and at 99% - 10.15 expected violations. The unconditional coverage test is also used to check whether the observed number of violations is statistically equal to the expected number of violations. The achieved results for all models and all data sets are summarized in table 3.4.

Table 3.4: Comparison of various models for estimating VaR

Data	Model	VR		Observed number of violations		Unconditional coverage test[1]	
Significance level α		95%	99%	95%	99%	95%	99%
Expected				51	10		
$data1$	$GARCH(0,1)$	1.222	1.970	62	20	H_0	H_1
	$EMWA_{\lambda=0.94}$	0.926	1.970	47	20	H_0	H_1
	Basic model	0.768	1.773	39	18	H_0	H_0
$data2$	$GARCH(1,0)$	1.025	1.478	52	15	H_0	H_0
	$EMWA_{\lambda=0.94}$	0.966	1.970	49	20	H_0	H_1
	Basic model	0.828	1.576	42	16	H_0	H_0
$data3$	$GARCH(1,0)$	1.340	1.872	68	19	H_1	H_0
	$EMWA_{\lambda=0.94}$	0.847	2.365	43	24	H_0	H_1
	Basic model	0.828	1.970	42	20	H_0	H_1
$data4$	$GARCH(0,1)$	1.299	2.461	66	25	H_1	H_1
	$EMWA_{\lambda=0.94}$	0.925	2.264	47	23	H_0	H_1
	Basic model	0.669	1.870	34	19	H_1	H_0
$data5$	$GARCH(1,0)$	0.374	0.984	19	10	H_1	H_0
	$EMWA_{\lambda=0.94}$	0.197	0.886	10	9	H_1	H_0
	Basic model	0.157	0.787	8	8	H_1	H_0
$data6$	$GARCH(0,1)$	1.161	2.362	59	24	H_0	H_1
	$EMWA_{\lambda=0.94}$	1.102	2.657	56	27	H_0	H_1
	Basic model	1.024	2.756	52	28	H_0	H_1

Table 3.4: Comparison of various models for estimating VaR

Data	Model	VR		Observed number of violations		Unconditional coverage test[1]	
Significance level α		95%	99%	95%	99%	95%	99%
$data7$	$GARCH(1,0)$	1.378	1.870	70	19	H_1	H_0
	$EMWA_{\lambda=0.94}$	1.142	2.657	58	27	H_0	H_1
	Basic model	0.965	2.657	49	27	H_0	H_1
$data8$	$GARCH(1,0)$	1.181	1.378	60	14	H_0	H_0
	$EMWA_{\lambda=0.94}$	0.807	1.870	41	19	H_0	H_0
	Basic model	0.689	1.673	35	17	H_1	H_0
$data9$	$GARCH(0,3)$	0.965	1.181	49	12	H_0	H_0
	$EMWA_{\lambda=0.94}$	0.728	2.362	37	24	H_1	H_1
	Basic model	0.610	2.165	31	22	H_1	H_1
Expected				46	9		
$data10$	$GARCH(0,1)$	1	3.222	46	29	H_0	H_1
	$EMWA_{\lambda=0.94}$	1.196	3.889	55	35	H_0	H_1
	Basic model	1.0435	3.667	48	33	H_0	H_1

A comparison of the models in the data set $data1$ showed that all models performed well. The EWMA model gives the best VR at a 95% significance level, while the Basic model is the best one for a 99% significance level. Based on the unconditional coverage test, only the Basic model gives a good estimation at both levels of significance. It can be concluded that the GARCH model slightly underestimates VaR. The Basic model should be considered the best one. The EWMA model has the best performance at the 95% significance level, but it underestimates VaR at 99% level.

[1] *The notation H_0 means that the unconditional coverage test fails to reject the null hypothesis, which indicates that the observed number of violations is statistically equal to the expected number of violations, and H_1 - the number of violations differs significantly from its expected value.*

In the data set $data2$, the GARCH and Basic models give good accuracy at both significance levels. The EWMA model performs well at a 95% significance level, but underestimates the VaR at 99% level. Comparing the performance of the GARCH and Basic models based on the results of VR, the GARCH model gives a better estimation than the Basic model.

The EWMA model in data set $data3$ gives similar results to the data set $data2$ - slightly underestimates the VaR at a 99%, significance level, but it performs quite well at the 95% level, while the GARCH model gives the worst results at a 95% level. Although the violation ratio of the GARCH model at the 99% significance level is rather high, the unconditional coverage test suggests that the observed and expected violations are statistically equal. The results show that the performance of the Basic model is very similar to that of the EWMA model.

In data set $data4$ at the 95% significance level, only the EWMA model gives a good estimation, while the GARCH model slightly underestimates VaR and the Basic model - overestimates it. The Basic model gives sufficient results at the significance level of 99%, but the GARCH and EWMA models have the worst performance.

All models in the data set $data5$ heavily overestimate VaR at the 95% level, but give very good results at 99%. This can be explained by the characteristics of the data set. The figure 4.4 shows that the data is sparse, price does not change at each time moment, but only with a few peaks once in a while. The models used do not capture this effect very well. The GARCH model works slightly better, but it is not enough to use the model for good VaR estimation.

In the data set $data6$, the results are inverted than for the data set $data5$ - all models estimate VaR at the 95% significance level well, but underestimate at 99% level.

The EWMA and Basic models have similar performance in the data set $data7$ - the models have performed well at the 95% level, but slightly underestimates the VaR at 99%. The GARCH model underperforms at the 95% significance level, but estimates the VaR well at the 99% level.

In the data set $data8$, the unconditional coverage test suggests that the number of observed violations using the EWMA and GARCH models is statistically equal to the expected number of observations, although the GARCH model gives better results than the EWMA model when comparing violation ratios. The Basic model also appears to be good at the 99% level, but overestimates the VaR at 95%.

The GARCH model seems to be the best in the data set $data9$ - the violation ratio values are close to 1 and the unconditional coverage test fails to reject the null hypothesis. The EWMA and Basic models appear to have significantly worst performance for both levels of significance.

In the data set $data10$ at a 95% significance level, all models estimate VaR very well - the violation ratios are very close to 1, but for 99% level, all models highly underestimate VaR.

In practice, the method can be considered applicable for estimating VaR if $VR \in [0.75; 1.25]$ for 95% significance level and $VR \in [0.4; 1.9]$ at 99% level. The recommended $[0.8, 1.2]$ interval is extended due to the relatively small number of expected violations. Limits of extended intervals found using quantiles of Binomial distribution, recalling fact that number of violations has Binomial distribution. However, using more statistical approach, a method can be can be considered applicable if the observed number of violations is statistically equal to the expected one, in other words, the unconditional coverage test fails to reject the null hypothesis. The models that provide an adequate estimation for each data set are gathered in the table 3.5.

As shown in the table 3.5, at a 95% significance level, the EWMA model performed the best and appeared to be good in 8 out of 10 cases, in 3 cases the EWMA model outperformed the other two models. The GARCH model performed well in 6 cases out of 10, in 4 cases the GARCH model outperformed the other two models. The Basic model appeared to be good in 6 out of 10 cases, outperforming GARCH and EWMA models in 2 cases. In 1 case of 10, none of the models considered were suitable for estimating VaR at 95% significance level.

Table 3.5: *Models marked with ✓ gives adequate VaR estimation, using the violation ratio and unconditional coverage test*

Data set	GARCH		EWMA		Basic		None of the considered	
	95%	99%	95%	99%	95%	99%	95%	99%
$data1$	✓		✓		✓	✓		
$data2$	✓	✓	✓		✓			
$data3$		✓	✓		✓			
$data4$			✓			✓		
$data5$		✓		✓		✓	✓	
$data6$	✓		✓		✓			✓
$data7$		✓	✓		✓			
$data8$	✓	✓	✓	✓		✓		
$data9$	✓	✓						
$data10$	✓		✓		✓			✓

At a 99% significance level, the GARCH model has shown the best performance and appeared to be adequate in 6 out of 10 cases, in all 4 cases the GARCH model outperforms the other two models. The Basic model performed well in 4 case out of 10, outperforming the GARCH and EWMA models in 3 cases. The EWMA model appeared to be good also in 2 out of 10 cases, but did not outperform the GARCH and Basic models. There are 2 cases out of 10 when none of the considered models was adequate for estimating VaR at 99% significance level.

The GARCH model does not handle well the outliers, it tends to overestimate them. The performance of the model also depends on the choice of observation window used for the estimation. If there are several volatility clusters in one window, the estimation appears to be less accurate. At a 99% significance level the GARCH model has the best precision of the considered models.

The EWMA model is easy to use and very flexible, the model has very good precision at the 95% significance level. As mentioned above (and also seen from empirical use), it captures cluster volatility and adapts the data well. Although the EWMA model is not suggested if very high accuracy is required.

The Basic model is the simplest of the three considered models, it is smoother than both previous ones. Despite its simple approach, the model gives quite good performance results.

It can be concluded that there is no single model that would estimate VaR well enough for all data sets. The decision on which model to choose should be based on the characteristics of the data. The basic model could be preferred if very high accuracy is required or the data structure is atypical (eg data set $data5$) and there are indications that a more complex model is unlikely to give better results. The EWMA model could be preferred if not very high precision is required and the volatility is clearly clustered. The GARCH model is preferred when the distribution of returns is clearly not normal and skewed and data do not contain much outliers.

4 Conclusions

Value at Risk is a basic risk measure used in the finance industry to measure market risk. VaR determines potential maximal loss over a fixed time period at a specified confidence level. The key to estimate VaR is to estimate the loss distribution. Three models are used to estimate volatility in the VaR model - Basic, EWMA and GARCH models.

Several sources state that comparing VaR estimations that are achieved using various models is a complex task, no unique measure is suggested, usually multiple comparison methods are used to evaluate the performance of models. In the thesis the violation process and the unconditional coverage test are used to compare different models.

The Basic model is the simplest one and is based on classical formulas. It can be considered as a benchmark for other models. It is smoother than the other considered models and despite its simple approach gives quite good performance results, especially at a 95% significance level.

The approach of the EWMA model is to use a regular MA type model and adjust larger weights to more recent observations and smaller to older ones. The EWMA model is easy to use because it contains only one parameter. It is very flexible, the model has the best performance at the 95% significance level. It captures cluster volatility and adapts to the data well. Although, the EWMA model is not suggested if very high precision is required.

The GARCH model is the most complex of the considered models, it requires choosing the number of coefficients needed to be included in the model as well as the estimation of various parameters. The GARCH model does not handle the outliers well, it tends to overestimate outliers. It also depends on the choice of observation window used for estimation. If there are several volatility clusters in one window, the estimation appears to be less accurate. The GARCH model has the best precision at a 99% significance level.

The empirical assessment of GARCH, EWMA and Basic models to evaluate VaR shows that there is no single model that outperforms the others, the performance of the model depends on the characteristics of the data and the significance level required.

For further analysis to improve the estimations of VaR, a deeper analysis of outliers could be done, adding them to the model as explanatory variables. Also, some more sophisticated models could be tried out, such as the FGARCH, GJR-GARCH or CAViaR model. There is not enough empirical and theoretical proofs that volatility models measure VaR the best [12], so some new approaches besides derivatives of the Basic model could be introduced.

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Appendix 1. Resulting graphs for all data sets

In this appendix are presented graphs of for data sets $data2$, $data3$, $data4$, $data5$, $data6$, $data7$, $data8$, $data9$ and $data10$, compariing the actual observed returns with the estimated VaR values for all models .

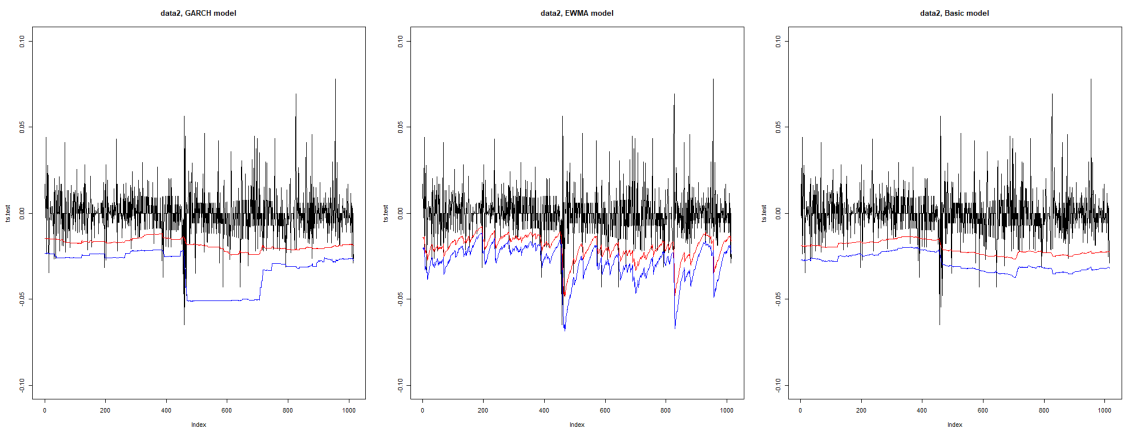

Figure 4.1: $data2$ *observed values of returns vs. estimated VaR. On the left - GARCH model, in the middle - EWMA model and on the right Basic model. Black line - observed values, red - estimated values at a* 95% *significance level and blue - at* 99% *significance level*

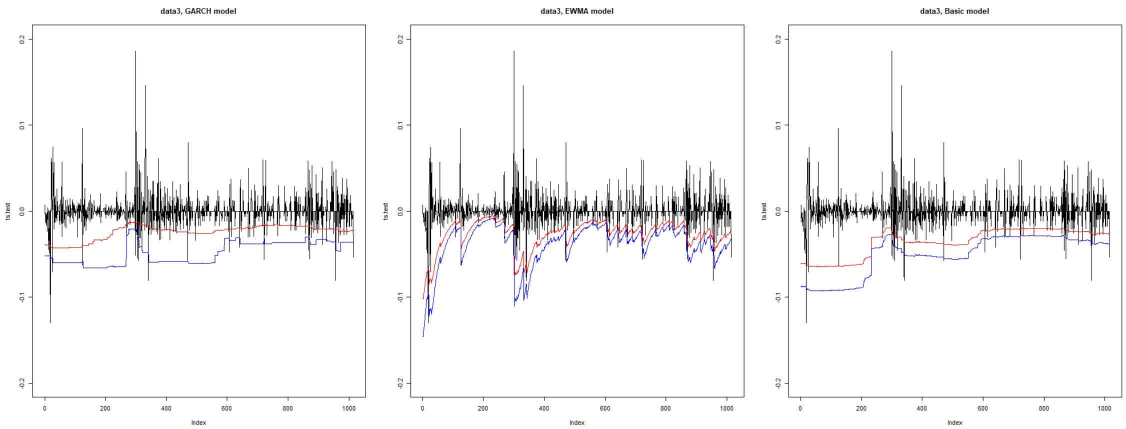

Figure 4.2: $data3$ *observed values of returns vs. estimated VaR. On the left - GARCH model, in the middle - EWMA model and on the right Basic model. Black line - observed values, red - estimated values at a* 95% *significance level and blue - at* 99% *significance level*

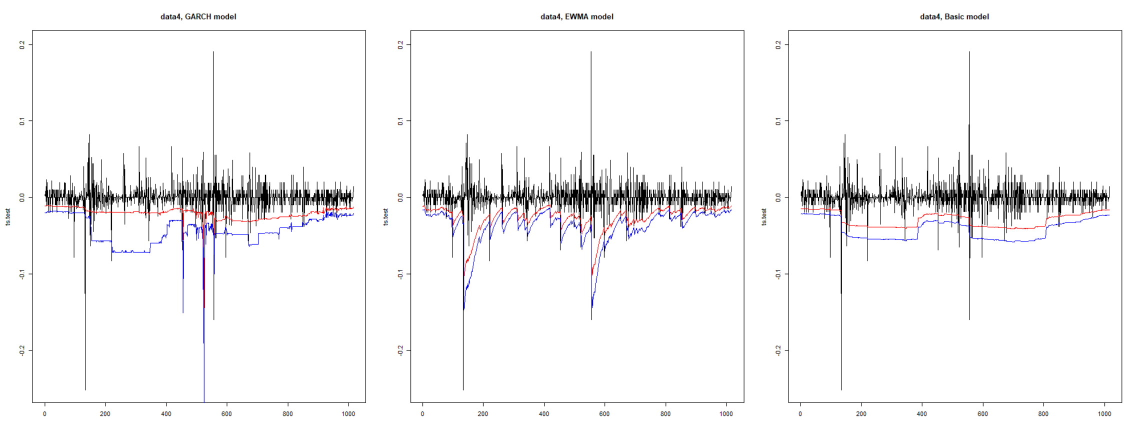

Figure 4.3: $data4$ *observed values of returns vs. estimated VaR. On the left - GARCH model, in the middle - EWMA model and on the right Basic model. Black line - observed values, red - estimated values at a* 95% *significance level and blue - at* 99% *significance level*

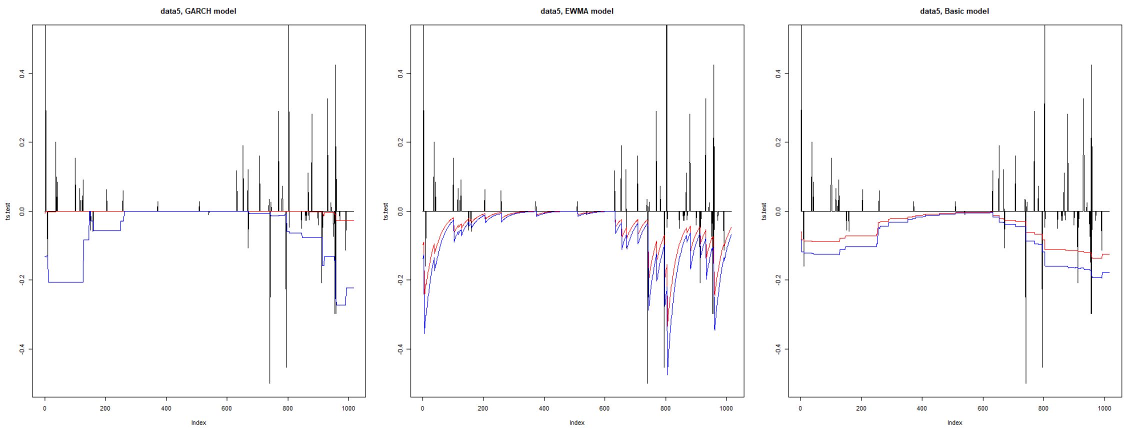

Figure 4.4: $data5$ *observed values of returns vs. estimated VaR. On the left - GARCH model, in the middle - EWMA model and on the right Basic model. Black line - observed values, red - estimated values at a* 95% *significance level and blue - at* 99% *significance level*

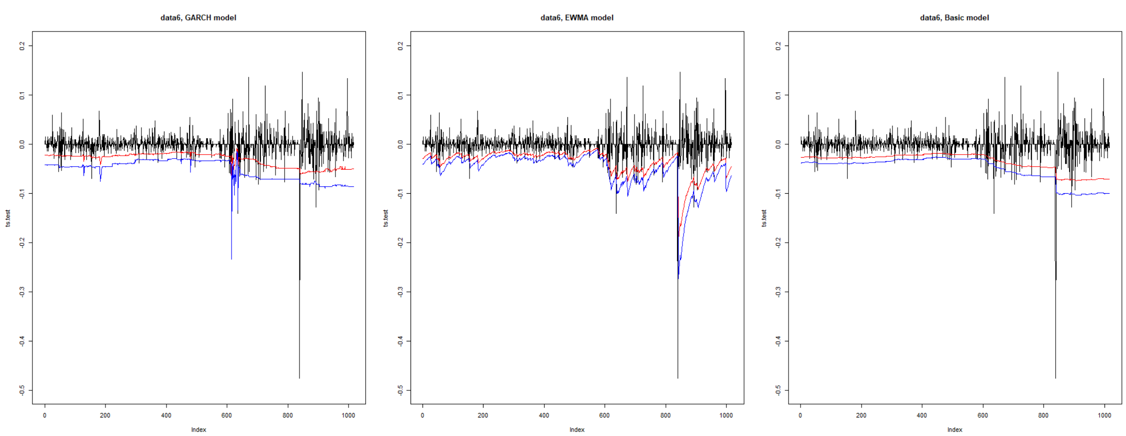

Figure 4.5: $data6$ *observed values of returns vs. estimated VaR. On the left - GARCH model, in the middle - EWMA model and on the right Basic model. Black line - observed values, red - estimated values at a* 95% *significance level and blue - at* 99% *significance level*

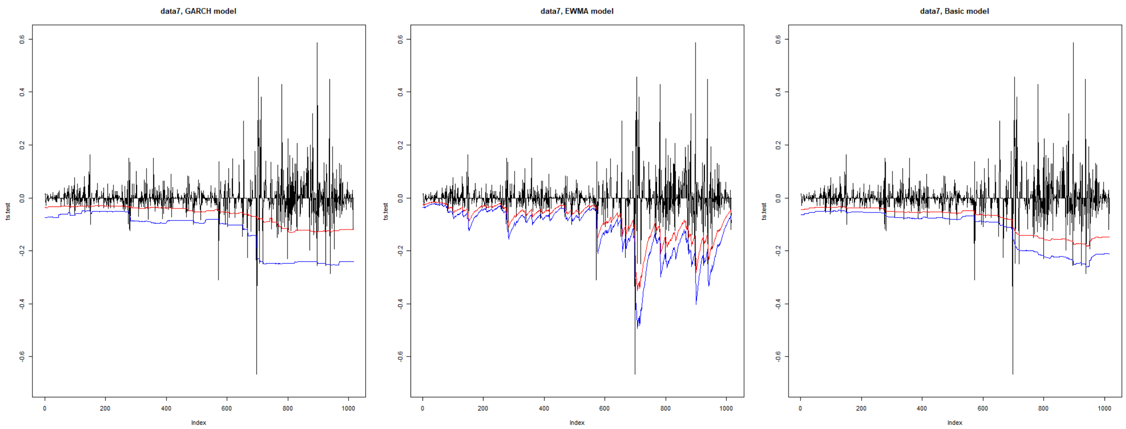

Figure 4.6: $data7$ *observed values of returns vs. estimated VaR. On the left - GARCH model, in the middle - EWMA model and on the right Basic model. Black line - observed values, red - estimated values at a* 95% *significance level and blue - at* 99% *significance level*

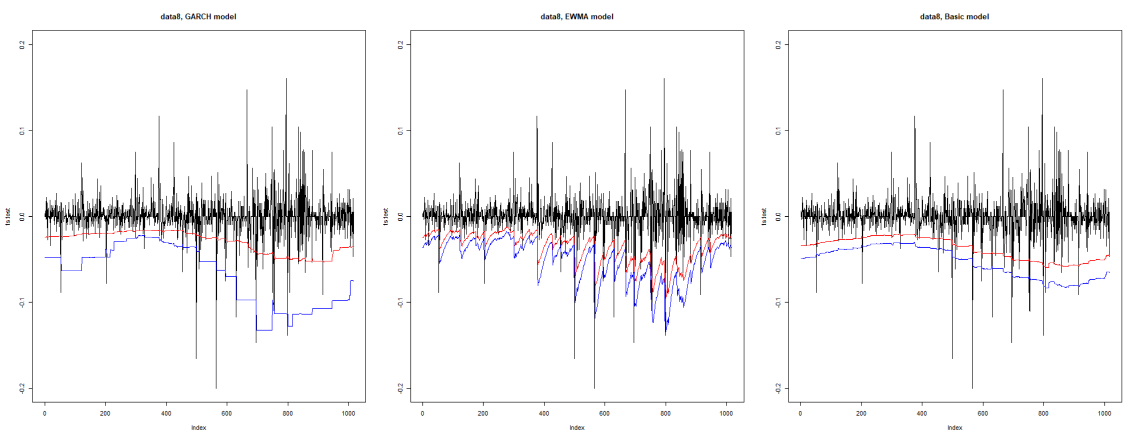

Figure 4.7: $data8$ *observed values of returns vs. estimated VaR. On the left - GARCH model, in the middle - EWMA model and on the right Basic model. Black line - observed values, red - estimated values at a* 95% *significance level and blue - at* 99% *significance level*

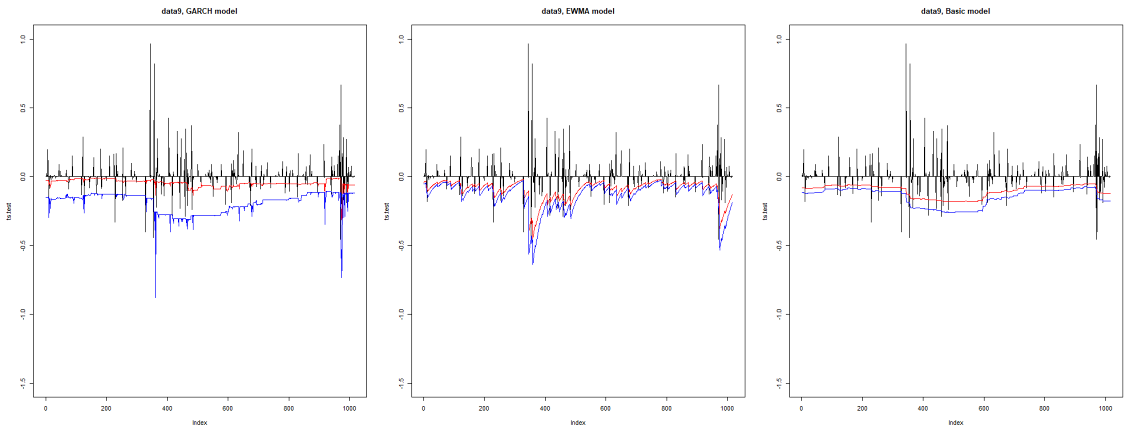

Figure 4.8: $data9$ *observed values of returns vs. estimated VaR. On the left - GARCH model, in the middle - EWMA model and on the right Basic model. Black line - observed values, red - estimated values at a* 95% *significance level and blue - at* 99% *significance level*

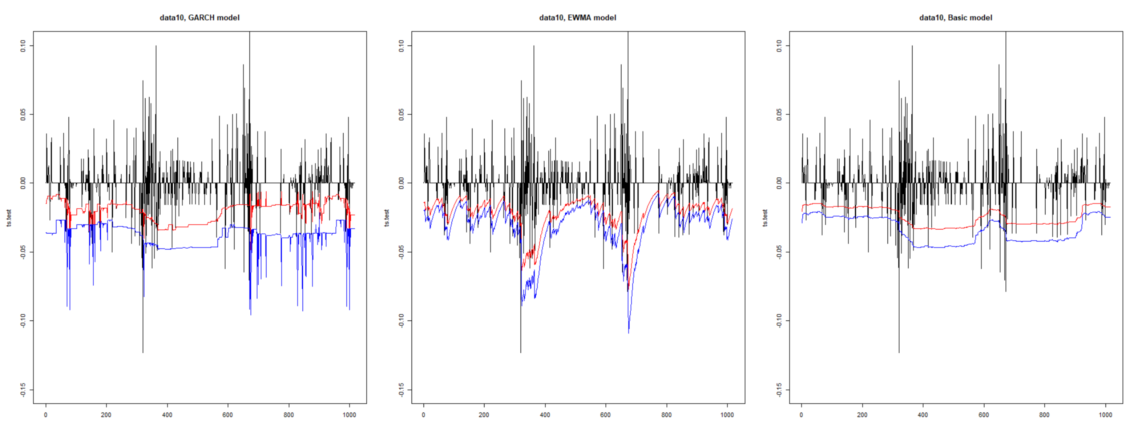

Figure 4.9: $data10$ *observed values of returns vs. estimated VaR. On the left - GARCH model, in the middle - EWMA model and on the right Basic model. Black line - observed values, red - estimated values at a* 95% *significance level and blue - at* 99% *significance level*

Appendix 2. R code

In this appendix are presented R code used in empirical part of this thesis.

```
#Libraries used in code
library(tseries)
library(Quandl)
library(expss)
library(rugarch)

#Function for plotting time series and its ACF and PACF.
tsgraphs1=function(s,lags=30){
  par(mfrow=c(1,2))
  acf(s,lag=lags)
  pacf(s,lag=lags)
  layout(1) }

#EWMA function
EWMA<-function(x,lambda) { #input values: x - time series, lambda - lambda value
  Variance=c()
  Variance[1]=x[1]*lambda
  Volatility=c()
  Volatility[1]=sqrt(Variance[1])
  pr=length(x)
  for (i in 2:pr){
    n=(1:i-1) ##this will be used for the weights
    z=as.matrix(n)
    w=c()
    w=(1-lambda)*lambda^z
    #arrange weights from least to greatest
    w=sort(w,decreasing=FALSE)
    product=w*x[1:i]
    ##multiply weights times squared log returns
    product=as.matrix(product) ##convert to matrix
    product=na.omit(product) ##remove all Nas in data
    Variance[i]=sum(product[1:i,]) ##sum the product
    Volatility[i]=sqrt(Variance[i])                     }
  final=cbind(Variance,Volatility)
  ##combine columns of Variance and Volatility                }

#VR function
VR1=function(data.test,var.value,alpha) {
  observed=sum(var.value<data.test)
  expected=alpha*(length(data.test))
  VR=observed/expected #Valioation ratio calculation
  final=cbind(observed,expected,VR)             }
```

```
#Data read in. Name  of the data compatabe with company name.
data1=read.table('grindex.csv',header=TRUE,sep=",")
data.price=rev(data1$Close.price) #Select data set
data=(data.price[2:length(data.price)]/data.price[1:(length(data.price)-1)])-1 #calculate returns
g=250 #length of window used to estimate model
ts.train=ts.full[1:g+1] #data window for first observation estimate
ts.test=ts.full[(g+2):1265] #the rest of the data

##GARCH model detection
#Plot graphs to evalute number of parameters in candid models
tsgraphs(ts.train,lags=45)
tsgraphs1(ts.train,lags=45)
#Candid ARMA models.
#Comparing diagnose plots and AIC, BIC values
m1.1=arima(ts.train,order=c(1,0,1)) #ARMA (1,1)
AIC(m1.1)
BIC(m1.1)
m1.2=arima(ts.train,order=c(1,0,0)) #AR(1)
AIC(m1.2)
BIC(m1.2)
m1.3=arima(ts.train,order=c(0,0,1)) #MA(1)
AIC(m1.3)
BIC(m1.3)
tsdiag(m1.3,60)
#GARCH model,using the same p and q values as for best ARMA model
res1=residuals(m1.3) #residuals of the best model
p=0
q=1
model.garch=garch(res1^2,order=c(p,q)) #GARCH model

##VaR estimation
l=length(ts.test) #number of observations need to be estimated
window.set=ts.train #first window
#vectors for storing VaR values for the Basic model
var5.lim=c()
var1.lim=c()
lambda=0.94 #lambda value
#vector for storing VaR values for the EWMA model
var5.ewma2=c()
var1.ewma2=c()
#vector for storing VaR values for the GARCH model
var5.garch=c()
var1.garch=c()
for (j in 1:l){
  m=length(window.set)
```

```
  window.set=window.set[(m-g):m] #window of j-th step
  mu=mean(window.set) #mean value within j-th window
  sigma=sd(window.set) #stanadrt deviation within j-th window
  results.ewma=EWMA(window.set^2,lambda=0.94) #the EWMA model
  sigma2=results.ewma[g,2] #volatility estimation using the EWMA model
  res=residuals(arima(window.set,order=c(p,0,q))) #residuals of ARMA model
  results.garch=garch(res^2,order=c(p,q),trace=FALSE) #the GARCH model
  sigma3=results.garch$fitted.values[g] #volatility estimation using the GARCH model
  x1=res[(p+q+1):g]/na.omit(results$fitted.values[1:g,1])#empirical distribution of \espilon
  #VaR estimation using Basic model at j-th step
  var5.lim[j]=mu-1.645*sigma
  var1.lim[j]=mu-2.326*sigma
  #VaR estimation using EWMA model at j-th step
  var5.ewma2[j]=mu-1.645*sigma2
  var1.ewma2[j]=mu-2.326*sigma2
   #VaR estimation using GARCH model at j-th step
  var5.garch[j]=mu-abs(quantile(x1,0.05))*sigma3
  var1.garch[j]=mu-abs(quantile(x1,0.01))*sigma3
  window.set=c(window.set,ts.test[j]) #moving window one step foreward
}

#Calculate VR and UCC test
vr.garch5=VR1(ts.test,var5.garch,0.05)
vr.garch1=VR1(ts.test,var1.garch,0.01)
VaRTest(alpha=0.05,ts.test,var5.garch,conf.level = 0.95)$uc.Decision
VaRTest(alpha=0.01,ts.test,var1.garch,conf.level = 0.99)$uc.Decision
vr.lim5=VR1(ts.test,var5.lim,0.05)
vr.lim1=VR1(ts.test,var1.lim,0.01)
VaRTest(alpha=0.05,ts.test,var5.lim,conf.level = 0.95)$uc.Decision
VaRTest(alpha=0.01,ts.test,var1.lim,conf.level = 0.99)$uc.Decision
vr.ewma2.5=VR1(ts.test,var5.ewma2,0.05)
vr.ewma2.1=VR1(ts.test,var1.ewma2,0.01)
VaRTest(alpha=0.05,ts.test,var5.ewma2,conf.level = 0.95)$uc.Decision
VaRTest(alpha=0.01,ts.test,var1.ewma2,conf.level = 0.99)$uc.Decision
```

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